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DIFFRACTION OF ELASTIC WAVES BY TWO COPLANAR GRIFFITH CRACKS IN AN INFINITE ELASTIC MEDIUM

D. L. JAIN and R. P. KANWAL

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania

Abstract—The problem of diffraction of normally incident longitudinal and antiplane shear waves by two parallel and coplanar Griffith cracks embedded in an infinite, isotropic and homogeneous elastic medium is solved. Approximate formulas are derived for the displacement field, stress tensor, stress intensity factors, far-field amplitudes and scattering cross section when the wave lengths are large compared to the distance between the outer edges of the two cracks. By taking appropriate limits we derive various interesting and new results. Furthermore, we derive the solution of the corresponding problem of diffraction of a plane acoustic wave by two rigid coplanar and parallel strips.

1. INTRODUCTION

THE problem of diffraction of elastic waves by various two and three-dimensional configurations of practical interest has attracted considerable attention. In this paper we present the solution of the problem of diffraction of normally incident longitudinal and antiplane shear waves by two symmetrical coplanar Griffith cracks located in an infinite, isotropic and homogeneous elastic medium. The faces of each of the cracks are assumed to be separated by a small distance so that, during small deformations of the solid, the crack faces do not come into contact. A simple integral equation technique enables us to obtain approximate values of the displacement field, stress tensor, stress intensity factors, far-field amplitudes and scattering cross section. In the elastostatic limit, we derive the value of the stress distribution in the neighborhood of two parallel and coplanar Griffith cracks which are opened by a constant pressure along the cracks. The limiting results so obtained agree with those obtained by Lowengrub and Srivastava [1].

By making the distance between the inner edges of the cracks tend to zero, we solve the diffraction problem for a single Griffith crack. The value of the stress intensity factor so derived agrees with the one given by Mal [2] while the formula for the scattering cross section even for this limiting case appears to be new. Furthermore, from our analysis we also obtain the formula for the scattering cross section of two perfectly rigid parallel and coplanar strips in acoustic diffraction.

Consider a rectangular cartesian coordinate system such that these cracks are located in the region $-a \leq x \leq -b$, $b \leq x \leq a$, $-\infty < y < \infty$, $z = 0$. It is convenient to normalize all lengths with respect to a which is half of the distance between the outer edges of these cracks. Then by setting $c = b/a$, the cracks are defined by equations $-1 \leq x \leq -c$, $c \leq x \leq 1$, $-\infty < y < \infty$, $z = 0$.

Let a plane harmonic elastic wave (the factor $e^{-i\omega t}$ is suppressed throughout the analysis) originating at $z = -\infty$ be incident normally on the cracks. This wave can be decomposed into a longitudinal (or P) wave and a shear (or S) wave. The displacement field in the P

wave is parallel to the direction of propagation which in the present case is the z-axis. The speed of P wave is $\{(\lambda+2\mu)/\rho_0\}^{\frac{1}{2}}$, where λ, μ are the Lamé constants and ρ_0 is the density of the medium. The S wave has a displacement field polarized in the plane perpendicular to the direction of propagation i.e. the plane $z = 0$ and its speed is $(\mu/\rho_0)^{\frac{1}{2}}$. The S wave can be further decomposed into SV and SH waves. They produce the displacement fields parallel to x-axis and y-axis, respectively. As is well known, the boundary value problems associated with each of the above three components of the incident field can be formulated independently of the other two. Since the diffraction problems relating to SV and P waves are similar in nature we shall consider only the SH wave.

The dimensionless numbers in this analysis are

$$m_1 = \left(\frac{\omega^2\rho_0 a^2}{\lambda+2\mu}\right)^{\frac{1}{2}}, \qquad m_2 = \left(\frac{\omega^2\rho_0 a^2}{\mu}\right)^{\frac{1}{2}}, \qquad \tau = \frac{m_1}{m_2}. \tag{1.1}$$

2. INCIDENT P WAVE

In this case the incident field $\mathbf{u}^{(0)}(x, z)$ is given as

$$\mathbf{u}^{(0)}(x, z) = im_1 A_0\, e^{im_1 z}\mathbf{e}_3, \tag{2.1}$$

where $\mathbf{e}_3$ is the unit vector along z-axis. Thus, the only non-vanishing components of the incident displacement and stress fields are

$$u_z^{(0)}(x, z) = im_1 A_0\, e^{im_1 z}, \qquad \tau_{zz}^{(0)} = p_0\, e^{im_1 z}, \tag{2.2}$$

where $p_0 = -\rho_0 a^2\omega^2 A_0$.

The scattered field $\mathbf{u}(x, z)$ can be represented completely in terms of two scalar potentials $\phi_j(x, z)$, $j = 1, 2$, such that

$$u_x(x, z) = \frac{\partial\phi_1}{\partial x} - \frac{\partial\phi_2}{\partial z}, \qquad u_y(x, z) = 0, \qquad u_z(x, z) = \frac{\partial\phi_1}{\partial z} + \frac{\partial\phi_2}{\partial x}, \tag{2.3}$$

and ϕ_1 and ϕ_2 satisfy the Helmholtz equations

$$\nabla^2\phi_j + m_j^2\phi_j = 0, \qquad j = 1, 2. \tag{2.4}$$

The values of the components of the stress tensor are

$$\tau_{xz} = \mu\left\{2\frac{\partial^2\phi_1}{\partial x\partial z} + \frac{\partial^2\phi_2}{\partial x^2} - \frac{\partial^2\phi_2}{\partial z^2}\right\}, \qquad \tau_{yz} = 0, \tag{2.5}$$

$$\tau_{zz} = -\mu\left\{\left(m_2^2 + 2\frac{\partial^2}{\partial x^2}\right)\phi_1 - 2\frac{\partial^2\phi_2}{\partial x\partial z}\right\}. \tag{2.6}$$

The boundary conditions are

$$\tau_{xz}(x, z) = 0, \qquad \tau_{zz}(x, z) + \tau_{zz}^{(0)}(x, z) = 0,\ z = 0, \qquad c \le |x| \le 1, \tag{2.7}$$

$$u_x,\ u_z,\ \tau_{xz} \ \text{ and } \ \tau_{zz} \text{ are continuous across } z = 0,\ |x| < c, \qquad |x| > 1. \tag{2.8}$$

In addition, we have to satisfy the radiation conditions at infinity and the edge conditions at the tips of the cracks.

We seek the solutions of the Helmholtz equations (2.4) in the form

$$\phi_1(x, z) = -i\int_{-\infty}^{\infty}\left(p^2-\frac{1}{2}m_2^2\right)\frac{P(p)}{\gamma_1}\,e^{ipx-\gamma_1|z|}\,dp, \tag{2.9}$$

and

$$\phi_2(x, z) = \mp\int_{-\infty}^{\infty} pP(p)\,e^{ipx-\gamma_2|z|}\,dp, \qquad z \gtrless 0, \tag{2.10}$$

where

$$\gamma_j = \begin{cases} (p^2-m_j^2)^{\frac{1}{2}}, \; p \ge m_j, \\ -i(m_j^2-p^2)^{\frac{1}{2}}, \; p \le m_j, \end{cases} \qquad j = 1, 2, \tag{2.11}$$

and where $P(p)$ is an unknown function to be determined from the remaining boundary conditions. Putting these values in the representation formulas (2.3), (2.5) and (2.6) we obtain

$$u_x(x, z) = \int_{-\infty}^{\infty} p\left[\frac{1}{\gamma_1}(p^2-\tfrac{1}{2}m_2^2)\,e^{-\gamma_1|z|}-\gamma_2\,e^{-\gamma_2|z|}\right]P(p)\,e^{ipx}\,dp, \tag{2.12}$$

$$u_z(x, z) = \pm i\int_{-\infty}^{\infty}[(p^2-\tfrac{1}{2}m_2^2)\,e^{-\gamma_1|z|}-p^2\,e^{-\gamma_2|z|}]P(p)\,e^{ipx}\,dp, \; z \gtrless 0, \tag{2.13}$$

$$\tau_{xz}(x, z) = \mp\mu\int_{-\infty}^{\infty} p[(2p^2-m_2^2)\,e^{-\gamma_1|z|}-(2p^2-m_2^2)\,e^{-\gamma_2|z|}]P(p)\,e^{ipx}\,dp, \; z \gtrless 0, \tag{2.14}$$

$$\tau_{zz}(x, z) = \mu i\int_{-\infty}^{\infty}\left[\frac{1}{\gamma_1}(p^2-\tfrac{1}{2}m_2^2)(m_2^2-2p^2)\,e^{-\gamma_1|z|}+2p^2\gamma_2\,e^{-\gamma_2|z|}\right]P(p)\,e^{ipx}\,dp. \tag{2.15}$$

With the help of the foregoing relations, the boundary conditions (2.7), (2.8) and the evenness of the function $P(p)$ yield the dual integral equations

$$\int_0^{\infty} P(p)\cos px\,dx = 0, |x| < c, \qquad x > 1, \tag{2.16}$$

and

$$\int_0^{\infty}\left[p^2\gamma_2-\frac{1}{\gamma_1}(p^2-\tfrac{1}{2}m_2^2)^2\right]P(p)\cos px\,dp = \frac{ip_0}{4\mu}, \qquad c \le |x| \le 1. \tag{2.17}$$

To solve them we set

$$P(p) = \frac{1}{p}\int_c^1 h(x_1^2)\sin px_1\,dx_1, \tag{2.18}$$

where the function $h(x_1^2)$ will be soon determined. For the interval $|x| > 1$, equation (2.16) is automatically satisfied in view of the formula

$$\int_0^{\infty}\frac{\sin px_1\cos px}{p}\,dp = \begin{cases} \frac{\pi}{2}, |x| < x_1, \\ 0, |x| > x_1. \end{cases} \tag{2.19}$$

Furthermore, from equations (2.16)–(2.19) we derive the following relations in $h(x_1^2)$:

$$\int_c^1 \int_0^\infty h(x_1^2) \sin px_1 \cos px \, dp \, dx_1 = \frac{ip_0}{2\mu(m_2^2 - m_1^2)} + \int_c^1 \int_0^\infty h(x_1^2)$$

$$\times \left\{1 - \frac{2}{(m_2^2 - m_1^2)p}\left[p^2\gamma_2 - \frac{1}{\gamma_1}(p^2 - \tfrac{1}{2}m_2^2)^2\right]\right\} \sin px_1 \cos px \, dp \, dx_1, \qquad c \le |x| \le 1, \tag{2.20}$$

and

$$\int_c^1 h(x_1^2) \, dx_1 = 0. \tag{2.21}$$

By using the relations

$$\frac{\sin px_1 \sin px}{p} = \frac{\pi}{2}(xx_1)^{\frac{1}{2}} J_{\frac{1}{2}}(px) J_{\frac{1}{2}}(px_1), \tag{2.22}$$

and

$$J_{\frac{1}{2}}(px) = \left(\frac{2p}{\pi x}\right)^{\frac{1}{2}} \int_0^x \frac{wJ_0(pw)\,dw}{(x^2 - w^2)^{\frac{1}{2}}}, \tag{2.23}$$

and after some simple manipulations, equation (2.20) can be written as the integro–differential equation:

$$\int_c^1 \frac{x_1 h(x_1^2)\,dx_1}{x_1^2 - x^2} = \frac{\pi}{2m_2^2}P_0 + \frac{d}{dx}\int_c^1 h(x_1^2) \int_0^x \int_0^{x_1} \frac{vwL_1(v, w)\,dv\,dw\,dx_1}{(x^2 - w^2)^{\frac{1}{2}}(x_1^2 - v^2)^{\frac{1}{2}}}, \qquad c \le x \le 1, \tag{2.24}$$

where

$$P_0 = ip_0/\pi\mu(1 - \tau^2), \qquad \tau = m_1/m_2$$

and

$$L_1(v, w) = \int_0^\infty \left\{p - \frac{2}{(m_2^2 - m_1^2)}\left[p^2\gamma_2 - \frac{1}{\gamma_1}(p^2 - \tfrac{1}{2}m_2^2)^2\right]\right\} J_0(pv)J_0(pw)\,dp. \tag{2.25}$$

Thus, the function $h(x_1^2)$ is to be determined from equations (2.24) and (2.21).

By using Noble's contour integration technique [3], the kernel $L_1(v, w)$ can be written as

$$L_1(v, w) = \frac{2im_2^2\tau^4}{1 - \tau^2}\int_0^1 \left\{\frac{1}{(1 - p^2)^{\frac{1}{2}}}\left(p^2 - \frac{1}{2\tau^2}\right)^2 J_0(pm_1 v)H_0^{(1)}(pm_1 w) + \frac{p^2(1 - p^2)^{\frac{1}{2}}}{\tau^4} J_0(pm_2 v)H_0^{(1)}(pm_2 w)\right\} dp, \qquad w > v. \tag{2.26}$$

The value of this kernel for $w < v$ is obtained by interchanging v and w. The form (2.26) is suitable for expanding $L_1(v, w)$ in powers of m_1 and m_2 such that $m_2 \ll 1$. Thus $m_1 \ll 1$, $m_1 = O(m_2)$. When we substitute the series expansions for the Bessel function J_0 and the Hankel function $H_0^{(1)}$, we get from (2.26)

$$L_1(v, w) = -[c_1 m_2^2 \log m_2 + (c_2 + c_1 \log v)m_2^2 + O(m_2^4 \log m_2)], \qquad v > w, \tag{2.27}$$

where

$$c_1 = \frac{(3\tau^4-4\tau^2+3)}{4(1-\tau^2)}, \tag{2.28}$$

$$c_2 = \frac{(3\tau^4-4\tau^2+2)}{4(1-\tau^2)}\log\tau+\frac{\pi}{2}\left(\frac{2\gamma}{\pi}-i\right)c_1-\frac{\log 2}{4(1-\tau^2)}(11\tau^4-12\tau^2+5)$$

$$+\frac{4}{\pi}[N_2(1+\tau^2)+\tau^2N_0], \tag{2.29}$$

$$N_{2n} = \int_0^1 p^{2n}(1-p^2)^{\frac{1}{2}}\log p\,\mathrm{d}p, \qquad n = 0, 1, 2, \ldots, \tag{2.30}$$

and γ is Euler's constant.

In particular

$$N_0 = -\frac{\pi}{8}(1+\log 4), \qquad N_2 = \frac{\pi}{64}(1-4\log 2). \tag{2.31}$$

Let us now expand $h(x_1^2)$ in the form

$$h(x_1^2) = \frac{1}{m_2^2}[h_0(x_1^2)+(m_2^2\log m_2)h_1(x_1^2)+m_2^2h_2(x_1^2)+m_2^4(\log m_2)^2h_3(x_1^2)+0(m_2^4\log m_2)], \tag{2.32}$$

and substitute this expansion as well as the expansion (2.27) of $L_1(v, w)$ in equation (2.24), equate the coefficients of equal powers of m_2. Thereby we get the following equations for determining the unknown functions $h_i(x_1^2)$, $i = 0, 2, 3, \ldots$:

$$\int_c^1 \frac{x_1h_0(x_1^2)\,\mathrm{d}x_1}{x_1^2-x^2} = \frac{\pi}{2}P_0, \qquad c \le |x| \le 1; \qquad \int_c^1 h_0(x_1^2)\,\mathrm{d}x_1 = 0, \tag{2.33}$$

$$\int_c^1 \frac{x_1h_1(x_1^2)\,\mathrm{d}x_1}{x_1^2-x^2} = -c_1\int_c^1 x_1h_0(x_1^2)\,\mathrm{d}x_1, \qquad c \le x \le 1; \qquad \int_c^1 h_1(x_1^2)\,\mathrm{d}x_1 = 0, \tag{2.34}$$

$$\int_c^1 \frac{x_1h_2(x_1^2)\,\mathrm{d}x_1}{x_1^2-x^2} = -c_1\frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 h_0(x_1^2)\int_0^x\int_0^{x_1}\frac{vw\begin{Bmatrix}\log v;\; v\ge w\\ \log w;\; w\ge v\end{Bmatrix}\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x_1}{(x^2-w^2)^{\frac{1}{2}}(x_1^2-v^2)^{\frac{1}{2}}}$$

$$-c_2\int_c^1 x_1h_0(x_1^2)\,\mathrm{d}x_1, \qquad c \le x \le 1; \qquad \int_c^1 h_2(x_1^2)\,\mathrm{d}x_1 = 0, \tag{2.35}$$

$$\int_c^1 \frac{x_1h_3(x_1^2)\,\mathrm{d}x_1}{(x_1^2-x^2)} = -c_1\int_c^1 x_1h_1(x_1^2)\,\mathrm{d}x_1, \qquad c \le x \le 1; \qquad \int_c^1 h_3(x_1^2)\,\mathrm{d}x_1 = 0, \tag{2.36}$$

and so on. It has been shown by Lowengrub and Srivastava [1] that the solution of the integral equation

$$\int_c^1 \frac{x_1h(x_1^2)\,\mathrm{d}x_1}{x_1^2-x^2} = \frac{\pi}{2}f(x), \qquad c \le |x| \le 1, \tag{2.37}$$

subject to the condition

$$\int_c^1 h(x_1^2)\,\mathrm{d}x_1 = 0, \tag{2.38}$$

where $f(x)$ is a known even function, is

$$h(x_1^2) = -\frac{2}{\pi}\left(\frac{x_1^2-c^2}{1-x_1^2}\right)^{\frac{1}{2}} \int_c^1 \left(\frac{1-x^2}{x^2-c^2}\right)^{\frac{1}{2}} \frac{xf(x)}{x^2-x_1^2}\,\mathrm{d}x + \frac{2}{\pi F(x_1^2-c^2)^{\frac{1}{2}}(1-x_1^2)^{\frac{1}{2}}} \int_c^1 \left(\frac{1-x^2}{x^2-c^2}\right)^{\frac{1}{2}} xf(x)\,\mathrm{d}x$$

$$\times \int_c^1 \left(\frac{x_1^2-c^2}{1-x_1^2}\right)^{\frac{1}{2}} \frac{\mathrm{d}x_1}{x^2-x_1^2}, \tag{2.39}$$

while $F = F(\pi/2, (1-c^2)^{\frac{1}{2}})$ is the elliptic integral of the first kind.

With the help of relations (2.37)–(2.39) we obtain the solutions of equations (2.33)–(2.36) as

$$h_0(x_1^2) = \frac{P_0[x_1^2 - E/F]}{\{(x_1^2-c^2)(1-x_1^2)\}^{\frac{1}{2}}}, \tag{2.40}$$

$$h_1(x_1^2) = -\frac{c_1P_0}{2}\,\frac{(1+c^2-2E/F)[x_1^2-E/F]}{\{(x_1^2-c^2)(1-x_1^2)\}^{\frac{1}{2}}}, \tag{2.41}$$

$$h_2(x_1^2) = P_0\left[\frac{d_0[x_1^2-E/F]-(c_1/2)[(x_1^2-c^2)(x_1^2-\frac{1}{2}(1-c^2))-(E/6)F(1+c^2)+(c^4/2-c^2/6)]}{\{(x_1^2-c^2)(1-x_1^2)\}^{\frac{1}{2}}}\right]$$

$$-P_0c_1\left[\frac{\begin{array}{l}(x_1^2-E/F)(1-E/F)+\{x_1^2(x_1^2-c^2)\}\\ \quad\times\{(1/F)\Pi(\pi/2,(1-c^2)/(1-x_1^2),(1-c^2)^{\frac{1}{2}})-1\}-(1-c^2)I/F^2\end{array}}{\{(x_1^2-c^2)(1-x_1^2)\}^{\frac{1}{2}}}\right] \tag{2.42}$$

$$h_3(x_1^2) = \frac{c_1^2P_0}{4}(1+c^2-2E/F)^2\left[\frac{(x_1^2-E/F)}{\{(x_1^2-c^2)(1-x_1^2)\}^{\frac{1}{2}}}\right], \tag{2.43}$$

where $E = E(\pi/2, (1-c^2)^{\frac{1}{2}})$ is the elliptic integral of the second kind and Π is the elliptic integral of the third kind

$$\Pi(\varphi, n^2, k) = \int_0^{\varphi} \frac{\mathrm{d}\alpha}{(1-n^2\sin^2\alpha)(1-k^2\sin^2\alpha)^{\frac{1}{2}}}$$

$$I = \int_0^{\pi/2} \sin^2\theta(\sin^2\theta + c^2\cos^2\theta)^{\frac{1}{2}}\{\Pi(\pi/2, \sec^2\theta, (1-c^2)^{\frac{1}{2}}) - F\}\,\mathrm{d}\theta,$$

and

$$d_0 = -\frac{c_2}{2}[1+c^2-2E/F] - \frac{c_1}{2}\left[(1+c^2)\log\left(\frac{1-c^2}{e}\right)^{\frac{1}{2}} - \frac{2E}{F}\log\frac{(1-c^2)^{\frac{1}{2}}}{e}\right].$$

Thus, the value of the required function $h(x_1^2)$ is known up to $0(m_2^4 \log m_2)$.

Let us now evaluate some quantities of physical interest.

Stress intensity factors

The stress intensity factors K_1 and K_2 are defined as (in physical units)

$$K_1 = \lim_{x\to 1+} (a)^{\frac{1}{2}}[(x-1)^{\frac{1}{2}}\{\tau_{zz}(x,0)\}]_{x>1}, \tag{2.44}$$

$$K_2 = \lim_{x\to c-} (a)^{\frac{1}{2}}[(c-x)^{\frac{1}{2}}\{\tau_{zz}(x,0)\}]_{x<c}. \tag{2.45}$$

The value of the stress component τ_{zz} can be evaluated from the formulas (2.15) and (2.18) when we substitute the value of the function $h(x_1^2)$ as obtained above. We spare the reader of the details. After evaluating the value of τ_{zz} and putting it in relations (2.44) and (2.45) we obtain

$$\begin{aligned} K_1 = \frac{p_0(a)^{\frac{1}{2}}}{\{2(1-c^2)\}^{\frac{1}{2}}}\Bigg\{&(1-E/F)-\frac{c_1}{1}(1+c^2-2E/F)(1-E/F)m_2^2\log m_2 \\ &+\left[d_0(1-E/F)-\frac{c_1}{2}\left\{\frac{1}{2}-\frac{c^2}{6}-\frac{E}{6F}(1+c^2)\right\}-c_1\left\{(1-E/F)^2-(1-c^2)\left(1+\frac{I}{F^2}\right)\right\}\right]m_2^2 \\ &+\frac{c_1^2}{4}(1+c^2-2E/F)^2(1-E/F)(m_2^2\log m_2)^2+0(m_2^4\log m_2)\Bigg\}, \end{aligned} \tag{2.46}$$

and

$$\begin{aligned} K_2 = \frac{p_0(a)^{\frac{1}{2}}}{[2c(1-c^2)]^{\frac{1}{2}}}\Bigg\{&(E/F-c^2)-\frac{c_1}{2}(1+c^2-2E/F)(E/F-c^2)m_2^2\log m_2 \\ &+\left[d_0(E/F-c^2)-\frac{c_1}{2}\left(\frac{E}{6F}(1+c^2)+\frac{c^2}{6}-\frac{c^4}{2}\right)-c_1\left\{(E/F-c^2)(1-E/F)+(1-c^2)\frac{I}{F^2}\right\}\right]m_2^2 \\ &+\frac{c_1^2}{4}(1+c^2-2E/F)^2(E/F-c^2)(m_2^2\log m_2)^2+0(m_2^4\log m_2)\Bigg\}. \end{aligned} \tag{2.47}$$

We have plotted the values of K_1 and K_2 in Figs. 1 and 2 (for $\tau^2 = 1/3$).

In the limit when $c \to 0$, we recover the stress intensity factor for one Griffith crack occupying the region $|x| \le a$, $-\infty < y < \infty, z = 0$:

$$K_1 = \frac{p_0\sqrt{a}}{\sqrt{2}}\left\{1-\frac{c_1}{2}m_2^2\log m_2-\frac{c_2}{2}m_2^2+\frac{c_1^2}{4}(m_2^2\log m_2)^2+0(m_2^4\log m_2)\right\}, \tag{2.48}$$

which agrees with the result of Mal [2].

On the other hand, when $\omega \to 0$ (the elastostatic limit), i.e. $m_1, m_2 \to 0$ and $A_0 \to -\infty$ such that p_0 tends to constant pressure P_1, relations (2.46) and (2.47) yield the stress intensity factors when two Griffith cracks are opened under constant pressure P_1. These limits are

$$K_1 = \frac{P_1 a^2}{\{2a(a^2-b^2)\}^{\frac{1}{2}}}(1-E/F); \qquad K_2 = \frac{P_1}{\{2b(a^2-b^2)\}^{\frac{1}{2}}}(a^2E/F-b^2), \tag{2.49}$$

which agree with the results given by Lowengrub and Srivastava [1].

Let us observe that by assuming the solution in the form (2.18) we have satisfied the edge condition as this relation implies that the stresses at the crack tips have the required

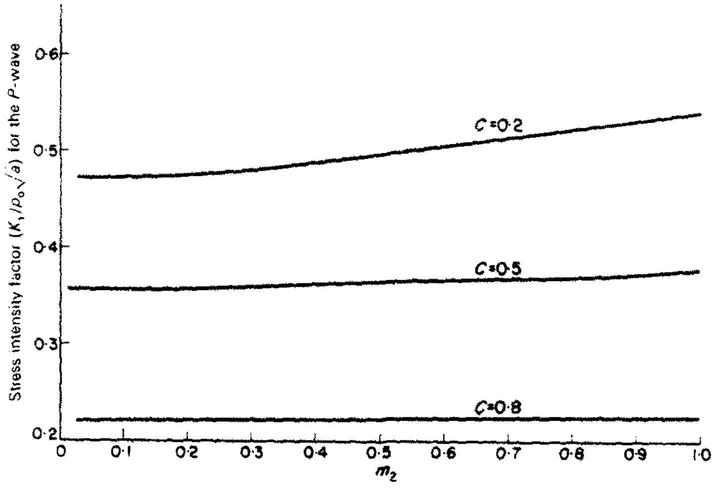

FIG. 1.

square root singularity. Indeed, if we calculate the value of the component u_z of the normal displacement at the face of the crack and use relation (2.20), we find that $u_z(x, 0\pm) = 0$, at $x = 1$ and $x = c$.

Far-field amplitude and scattering cross section

It is convenient to introduce polar coordinates.

$$x = R\sin\theta, \qquad z = R\cos\theta. \tag{2.50}$$

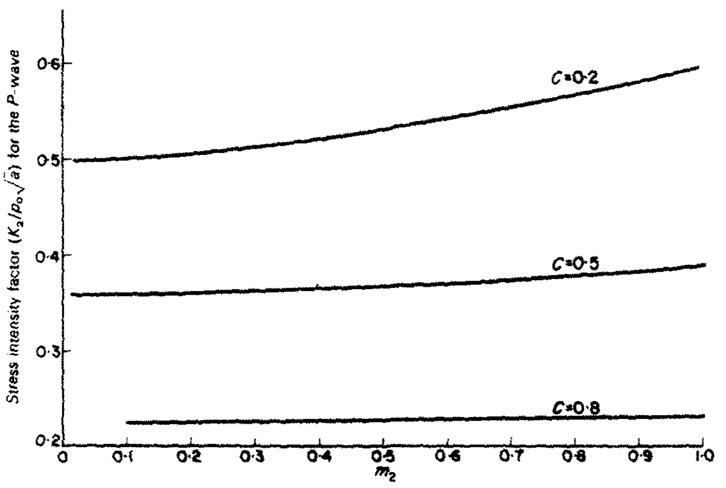

FIG. 2.

Then from relations (2.9) and (2.10) we readily derive, by the method of steepest descent, the asymptotic formulas

$$\phi_1(R,\theta) \sim \frac{\exp i(m_1R-\pi/4)}{(m_1R)^{\frac{1}{2}}}\left\{(2\pi)^{\frac{1}{2}}m_1^2\left(\sin^2\theta-\frac{1}{2\tau^2}\right)P(m_1\sin\theta)\right\}, \tag{2.51}$$

and

$$\phi_2(R,\theta) \sim \frac{\exp i(m_2R-\pi/4)}{(m_2R)^{\frac{1}{2}}}\{(2\pi)^{\frac{1}{2}}(m_2^2\sin\theta\cos\theta)P(m_2\sin\theta)\}, \tag{2.52}$$

as $R\to\infty$. Hence, the asymptotic values of the displacements $u_R(R,\theta)$ and $u_\theta(R,\theta)$ follow by putting the above values in relations (2.3). The results are

$$u_R(R,\theta) \sim \frac{i\exp i(m_1R-\pi/4)}{(m_1R)^{\frac{1}{2}}}(2\pi)^{\frac{1}{2}}m_1^3\left(\sin^2\theta-\frac{1}{2\tau^2}\right)P(m_1\sin\theta), \tag{2.53}$$

$$u_\theta(R,\theta) \sim \frac{i\exp i(m_2R-\pi/4)}{(m_2R)^{\frac{1}{2}}}(2\pi)^{\frac{1}{2}}m_2^3\sin\theta\cos\theta\, P(m_2\sin\theta), \tag{2.54}$$

as $R\to\infty$, where

$$\begin{aligned}P(m_j\sin\theta) = \frac{-iA_0}{4(1-\tau^2)}\Bigg[&(1+c^2-2E/F)-\tfrac{1}{2}c_1(1+c^2-2E/F)^2m_2^2\log m_2\\ &+m_2^2\left\{d_0(1+c^2-2E/F)-\frac{c_1}{2}\left(\frac{1}{4}+\frac{c^2}{6}-\frac{E}{3F}(1+c^2)\right)\right.\\ &\left.-c_1\left[(1+c^2-2E/F)(1-E/F)-2(1-c^2)\frac{I}{F^2}+\frac{1}{3}((1+c^2)E/F-(3-c^2))\right]\right\}\\ &+\frac{c_1^2}{4}(1+c^2-2E/F)^3(m_2^2\log m_2)^2-\frac{m_j^2}{6}\sin^2\theta\left(\frac{3}{4}+\frac{c^2}{2}+\frac{3}{4}c^4-\frac{E}{F}(1+c^2)\right)\\ &+O(m_2^4\log m_2)\Bigg],\ j=1,2.\end{aligned} \tag{2.55}$$

Hence, the radiation condition is satisfied. Writing relation (2.53) as

$$u_R(R,\theta) \sim \left(\frac{2}{\pi m_1R}\right)^{\frac{1}{2}}\left(\exp i\left(m_1R-\frac{\pi}{4}\right)\right)g(\theta),$$

we have the value of the scattering cross section [4],

$$\begin{aligned}\sum_p &= -\frac{4a}{m_1^2A_0}\mathscr{I}(g(0))\\ &= \frac{\pi^2am_2^3(3\tau^4-4\tau^2+3)}{32\tau(1-\tau^2)^2}[(1+c^2-2E/F)^2+O(m_2^2\log m_2)].\end{aligned} \tag{2.56}$$

In the limit when $c\to 0$, we get the corresponding value for one Griffith crack

$$\sum_p = \frac{\pi^2am_2^3(3\tau^4-4\tau^2+3)}{32\tau(1-\tau^2)^2}[1+O(m_2^2\log m_2)]. \tag{2.57}$$

As far as the authors are aware even this limiting formula is new. We have plotted the formula (2.56) for the value of $(\sum_p/a)$ in Fig. 3.

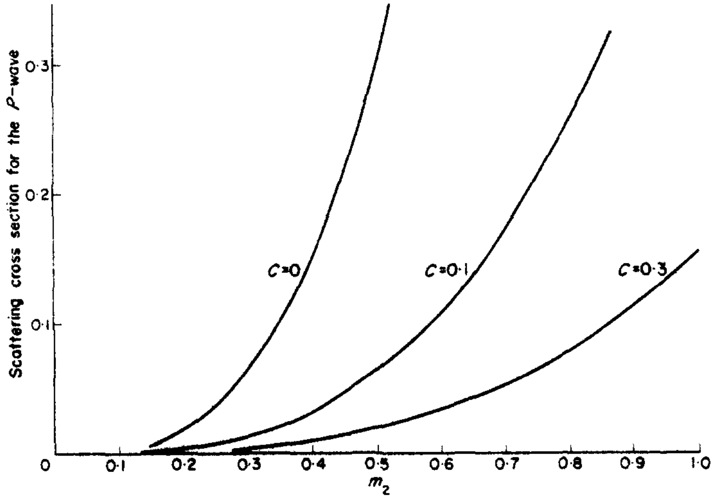

FIG. 3.

3. INCIDENT WAVE IS *SH* WAVE

For an incident *SH* wave we have

$$\mathbf{u}^{(0)}(x, z) = iB_0\, e^{im_2 z}\mathbf{e}_2, \tag{3.1}$$

where $\mathbf{e}_2$ is the unit vector along the y-axis. Thus, the only non-vanishing components of the incident field and the corresponding stress tensor are

$$u_y^{(0)}(x, z) = iB_0\, e^{im_2 z}, \qquad \tau_{yz}^{(0)} = q_0\, e^{im_2 z}, \tag{3.2}$$

where $q_0 = -\mu B_0 m_2$. Consequently, the displacement vector and the corresponding stress tensor due to the scattered field have, respectively, u_y and τ_{yz} as the only non-vanishing components needed in the sequel. As in the previous section we assume their values as

$$u_y(x, z) = \mp \int_{-\infty}^{\infty} Q(p)\, e^{ipx - \gamma_2 |z|}\, dp, \qquad z \lessgtr 0, \tag{3.3}$$

and

$$\tau_{yz} = \mu \int_{-\infty}^{\infty} \gamma_2 Q(p)\, e^{ipx - \gamma_2 |z|}\, dp, \tag{3.4}$$

where γ_2 is defined as in (2.11).

The boundary conditions for this problem are

$$\left.\begin{array}{ll} \tau_{yz} + \tau_{yz}^{(0)} = 0, \quad z = 0, c \le |x| \le 1, & \\ u_y \;\text{ and }\; \tau_{yz} \text{ are continuous across } z = 0, & |x| < c, |x| > 1. \end{array}\right\} \tag{3.5}$$

In addition, the radiation condition and the edge condition are to be satisfied. Relations (3.4) and $(3.5)_1$ yield

$$2\mu \int_0^{\infty} \gamma_2 Q(p) \cos px\, dp = -q_0, \qquad c \le x \le 1, \tag{3.6}$$

while from (3.3) and (3.5) we obtain

$$\int_0^\infty Q(p)\cos px\,\mathrm{d}p = 0, \qquad 0 \le |x| < c, \qquad |x| > 1,$$

where we have used the even character of the unknown function $Q(p)$.

As in the previous section we set

$$Q(p) = \frac{1}{p}\int_c^1 g(x_1^2)\sin px_1\,\mathrm{d}x_1, \tag{3.7}$$

and follow the same steps which led us to the integro–differential equation (2.24). The corresponding equation for the present problem is

$$\int_c^1 \frac{x_1 g(x_1^2)\,\mathrm{d}x_1}{(x_1^2-x^2)} = \frac{\pi}{2}Q_0 m_2 + \frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 g(x_1^2)\int_0^x\int_0^{x_1} \frac{wvL_2(v,w)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x_1}{(x^2-w^2)^{\frac{1}{2}}(x_1^2-v^2)^{\frac{1}{2}}}, \qquad c \le |x| \le 1, \tag{3.8}$$

subject to the condition

$$\int_c^1 g(x_1^2)\,\mathrm{d}x_1 = 0, \tag{3.9}$$

where $Q_0 = -q_0/(\mu\pi m_2) = B_0/\pi$, and

$$L_2(v,w) = \int_0^\infty [p-\gamma_2]J_0(pv)J_0(pw)\,\mathrm{d}p$$

$$= im_2^2\int_0^1 (1-p^2)^{\frac{1}{2}}J_0(pm_2w)H_0^{(1)}(pm_2v)\,\mathrm{d}p, \qquad v > w, \tag{3.10}$$

with v and w interchanged when $w > v$. In this case also, our analysis is based on the assumption $m_2 \ll 1$; and the expansion of $L_2(v,w)$ is

$$L_2(v,w) = -\tfrac{1}{4}m_2^2\log m_2 + (e_1 - \tfrac{1}{2}\log v)m_2^2 + O(m_2^4\log m_2), \qquad v > w, \tag{3.11}$$

where

$$e_1 = \tfrac{1}{4}(1 + i\pi - 2\gamma + 4\log 2).$$

Now we put the expansion

$$g(x_1^2) = m_2 g_0(x_1^2) + m_2^3\log m_2 g_1(x_1^2) + m_2^3 g_2(x_1^2) + m_2(m_2^2\log m_2)^2 g_3(x_1^2) + O(m_2^5\log m_2), \tag{3.12}$$

and the expansion (3.11) of $L_2(v,w)$ in equations (3.8) and (3.9) and equate the coefficients of equal powers of m_2. The resulting integral equations in functions g_0, g_1, g_2, g_3, can be solved in a fashion similar to equations (2.33)–(2.36). These solutions are

$$g_0(x_1^2) = \frac{Q_0(x_1^2-E/F)}{\{(x_1^2-c^2)(1-x_1^2)\}^{\frac{1}{2}}}, \tag{3.13}$$

$$g_1(x_1^2) = -\frac{Q_0}{4}(1+c^2-2E/F)\frac{(x_1^2-E/F)}{\{(x_1^2-c^2)(1-x_1^2)\}^{\frac{1}{2}}}, \tag{3.14}$$

$$g_2(x_1^2) = Q_0\left[\frac{e_0(x_1^2-E/F)-\frac{1}{4}[(x_1^2-c^2)(x_1^2-\frac{1}{2}(1-c^2))-(E/6F)(1+c^2)+(c^4/2-c^2/6)]}{\{(x_1^2-c^2)(1-x_1^2)\}^{\frac{1}{2}}}\right]$$

$$-\frac{Q_0}{2}\left[\frac{\begin{array}{l}(x_1^2-E/F)(1-E/F)+\{x_1^2(x_1^2-c^2)\}\\ \quad\times\{(1/F)\Pi(\pi/2,(1-c^2)/(1-x_1^2),(1-c^2)^{\frac{1}{2}})-1\}-(1-c^2)I/F^2\end{array}}{\{(x_1^2-c^2)(1-x_1^2)\}^{\frac{1}{2}}}\right], \tag{3.15}$$

$$g_3(x_1^2) = \frac{Q_0}{16}(1+c^2-2E/F)^2\frac{(x_1^2-E/F)}{\{(x_1^2-c^2)(1-x_1^2)\}^{\frac{1}{2}}} \tag{3.16}$$

where

$$e_0 = \frac{e_1}{2}(1+c^2-2E/F)-\frac{1}{4}\left[(1+c^2)\log\left(\frac{1-c^2}{e}\right)^{\frac{1}{2}}-\frac{2E}{F}\log\frac{(1-c^2)^{\frac{1}{2}}}{e}\right].$$

Thus, we have obtained the value of the unknown function $g(x_1^2)$ up to $0(m_2^5\log m_2)$.

Stress intensity factors

Using the foregoing relations in (3.4), we obtain

$$[\tau_{yz}(x,0)]_{\substack{x>1\\0\le x<c}} = 2\mu\int_c^1\frac{x_1 g(x_1^2)\,dx_1}{x_1^2-x^2}-2\mu\int_c^1 g(x_1^2)\int_0^x\int_0^{x_1}\frac{wvL_2(v,w)\,dv\,dw\,dx_1}{(x^2-w^2)^{\frac{1}{2}}(x_1^2-v^2)^{\frac{1}{2}}}. \tag{3.17}$$

Then the stress intensity factors (in physical units) are

$$\begin{aligned}K_1 &= \lim_{x\to 1+0}(a)^{\frac{1}{2}}\{(x-1)^{\frac{1}{2}}[\tau_{yz}(x,0)]_{x>1}\}\\ &= \frac{q_0(a)^{\frac{1}{2}}}{\{2(1-c^2)\}^{\frac{1}{2}}}\Bigg\{(1-E/F)-\frac{1}{4}(1+c^2-2E/F)(1-E/F)m_2^2\log m_2\\ &\quad+\left[e_0(1-E/F)-\frac{1}{4}\left\{\frac{1}{2}-\frac{c^2}{6}-\frac{E}{6F}(1+c^2)\right\}-\frac{1}{2}\left\{(1-E/F)^2-(1-c^2)\left(1+\frac{I}{F^2}\right)\right\}\right]m_2^2\\ &\quad+\frac{1}{16}(1+c^2-2E/F)^2(1-E/F)(m_2^2\log m_2)^2+0(m_2^4\log m_2)\Bigg\},\end{aligned} \tag{3.18}$$

$$\begin{aligned}K_2 &= \lim_{x\to c-}(a)^{\frac{1}{2}}\{(c-x)^{\frac{1}{2}}[\tau_{yz}(x,0)]_{x<c}\},\\ &= \frac{q_0(a)^{\frac{1}{2}}}{\{2c(1-c^2)\}^{\frac{1}{2}}}\Bigg\{(E/F-c^2)-\frac{1}{4}(1+c^2-2E/F)(E/F-c^2)m_2^2\log m_2\\ &\quad+\left[e_0(E/F-c^2)-\frac{1}{4}\left\{\frac{E}{6F}(1+c^2)+\frac{c^2}{6}-\frac{c^4}{2}\right\}-\frac{1}{2}\left\{(E/F-c^2)(1-E/F)+(1-c^2)\frac{I}{F^2}\right\}\right]m_2^2\\ &\quad+\frac{1}{16}(1+c^2-2E/F)^2(E/F-c^2)(m_2^2\log m_2)^2+0(m_2^4\log m_2)\Bigg\}.\end{aligned} \tag{3.19}$$

These values of K_1 and K_2 are plotted in Figs. 4 and 5 (for $\tau^2 = 1/3$).

When $c \to 0$, we recover the stress intensity factor for a single crack which agrees with Mal's result [2].

On the other hand in the elastostatic limit, when $\omega \to 0$ i.e. $m_1, m_2 \to 0$ and $B_0 \to -\infty$ such that $q_0 \to Q_1$, relations (3.18) and (3.19) yield the stress intensity factors when the

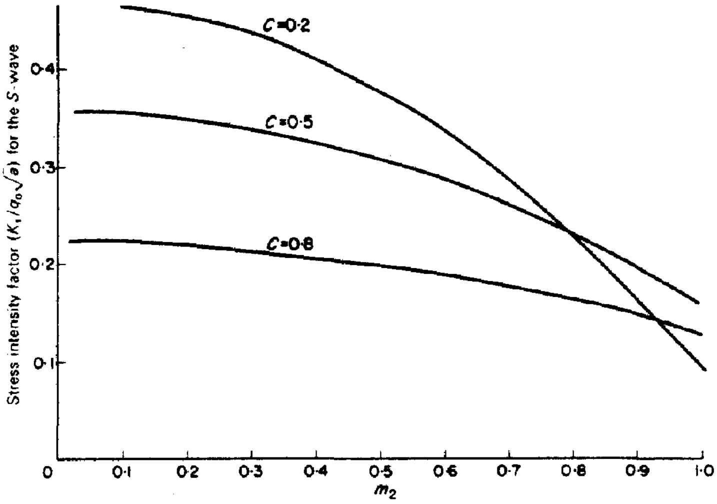

FIG. 4.

faces of two Griffith cracks are subjected to prescribed constant shearing stress, namely

$$\tau_{yz}(x, z) = -Q_1, \qquad z = 0, \qquad c \leq |x| \leq 1.$$

Thus in this case,

$$K_1 = \frac{Q_1 a^2}{\{2a(a^2-b^2)\}^{\frac{1}{2}}}(1-E/F); \; K_2 = \frac{Q_1}{\{2b(a^2-b^2)\}^{\frac{1}{2}}}(a^2E/F-b^2). \tag{3.20}$$

As pointed out in the case of P-wave, we have satisfied the edge condition by assuming the solution in the form (3.7).

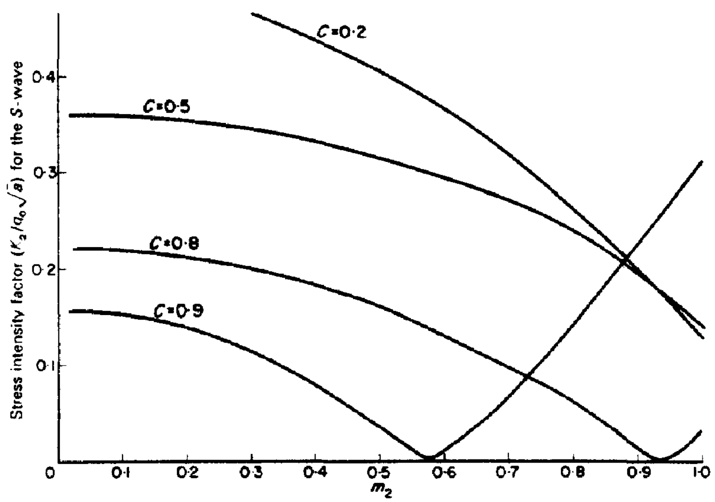

FIG. 5.

Far-field amplitude and scattering cross section

To find the far-field amplitude we proceed as in Section 2 and find that

$$u_y(R,\theta) \sim \left(\frac{2}{\pi m_2 R}\right)^{\frac{1}{2}} \{\exp i(m_2R-\pi/4)\}j(\theta) \quad \text{as} \quad R \to \infty, \tag{3.21}$$

where

$$\begin{aligned} j(\theta) = {} & -\frac{\pi}{4}B_0 m_2^2 \cos\theta \Bigg\{(1+c^2-2E/F)-\frac{1}{4}(1+c^2-2E/F)^2 m_2^2 \log m_2 \\ & + m_2^2\Bigg[e_0(1+c^2-2E/F)-\frac{1}{4}\left(\frac{1}{4}+\frac{c^2}{6}+\frac{1}{4}c^4-\frac{E}{3F}(1+c^2)\right)-\frac{1}{2} \\ & \times\left\{(1+c^2-2E/F)(1-E/F)-2(1-c^2)\frac{I}{F^2}+\frac{1}{3}[(1+c^2)E/F-(3-c^2)]\right\} \\ & -\frac{\sin^2\theta}{6}\left(\frac{3}{4}+\frac{c^2}{2}+\frac{3}{4}c^4-\frac{E}{F}(1+c^2)\right)\Bigg] \\ & +\frac{1}{16}(1+c^2-2E/F)^3(m_2^2\log m_2)^2+0(m_2^4\log m_2)\Bigg\}. \end{aligned} \tag{3.22}$$

Thus the radiation condition is satisfied. From (3.21) and (3.22) we have the value of the scattering cross section (in physical units) [4],

$$\begin{aligned} \textstyle\sum_{sh} &= -\frac{4a}{m_2 B_0}\mathscr{I}(j(0)) \\ &= \frac{\pi^2 a m_2^3}{8}[(1+c^2-2E/F)^2+0(m_2^2\log m_2)]. \end{aligned} \tag{3.23}$$

The value of $(\sum_{sh}/a)$ is plotted in Fig. 6 (for $\tau^2 = 1/3$).

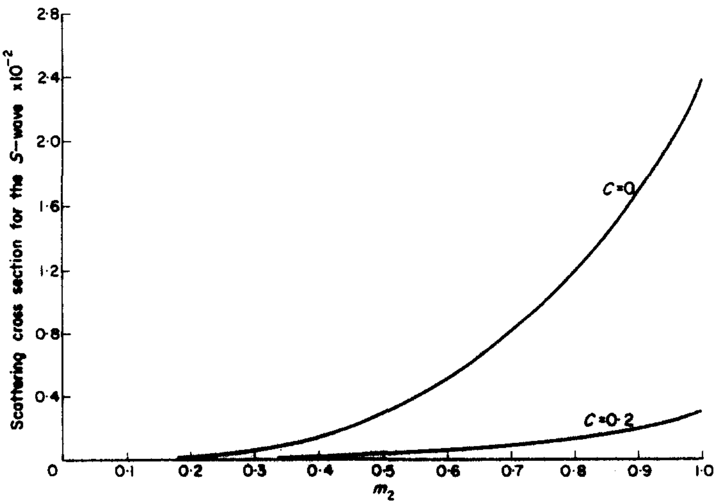

FIG. 6.

The results (3.21)–(3.23) are mathematically identical with those for the corresponding problem of diffraction of acoustic plane wave by two co-planar and parallel perfectly rigid strips or by two parallel slits in an infinite soft screen. In the limit when $c \to 0$, results (3.21)–(3.23) agree with the known results for the problem of diffraction of an acoustic plane wave by a perfectly rigid strip [5] or by a slit in an infinite soft screen [6]. This serves as another check on the above results.

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Абстракт—Решается задача дифракции нормально ударяющих продольных и антиплоских воин сдвига, вызванных двумя паралельными и компланарными трещинами Гриффитса, погруженными в бесконечной, изотропной, упругой среде. Определяются приближенные формулы для поля перемещкний тензора напряжений, факторов интенсивности напряжений, амплитуд для далёких полей и рассеяния в поперёчных сечениях, для-случая, когда длинц волн являются большими по сравнению с растоянием двух внешних краев двух трещин. Используя приближенные пределы получаются разные интересные и новые результаты. Далее определяется решение соотвествующей задачи дифракции для плоской, акустической волны, вызванной двумя жесткими и компланарными полосами.